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Covariant Casimir calculations

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Abstract. The Casimir effect is discussed and calculated using covariant methods. Ford's results on vacuum energy in curved space are considered and criticized from the point of view of non-covariance. The relevance of the 'conformal anomaly' of Fulling and Davies is noted.

1. Introduction

The Casimir effect (Casimir 1948, Boyer 1970) is well known. Most calculations of the vacuum energy density take a non-covariant form in that some sort of frequency cut-off is employed at an intermediate state. The neatest such derivation seems to be that of Fierz (1960). It has indeed been implied (DeWitt 1975, p 304) that covariance and covariant arguments will not give the Casimir energy (see the footnote in DeWitt, 1975). The explanation, of course, is not that covariance arguments are wrong but that they are incorrectly applied.

Ford (1975) has discussed quantum vacuum energy in a curved space background. His result for the De Sitter universe differs from that calculated by ourselves (Dowker and Critchley, 1976) and we wish to consider this fact in the present work. Firstly we consider the Casimir effect in flat space and give a covariant derivation. Then we turn to curved spaces.

2. The Casimir effect in flat space

The original Casimir effect concerned an electromagnetic field confined between parallel-plate conductors. We consider the same geometry but a massless scalar field, ϕ, like Ford (1975) and DeWitt (1975). The boundary conditions are either periodic ones or that ϕ vanishes on the plates.

We can write the vacuum energy–momentum tensor as the coincidence limit

$$\langle T_{\mu\nu}(x)\rangle = -\mathrm{i}\lim_{x'\to x}[(1-2\xi)\nabla_\mu\nabla_{\nu'}+g_{\mu\nu'}(2\xi-\tfrac{1}{2})g^{\lambda\sigma'}\nabla_\lambda\nabla_{\sigma'}-\xi(g_{\mu\rho'}\nabla^{\rho'}\nabla_{\nu'}+g_{\nu'\sigma}\nabla^\sigma\nabla_\mu)$$
$$+\xi g_{\mu\nu'}(\nabla_\rho\nabla^\rho+\nabla_{\rho'}\nabla^{\rho'})-\tfrac{1}{2}\xi(R_\mu^{\cdot\sigma}g_{\sigma\nu'}+g_{\mu\rho'}R^{\rho'}_{\cdot\nu'})+\tfrac{1}{2}\xi R g_{\mu\nu'}]D_{\mathcal{M}}(x,x') \qquad (1)$$

where $D_{\mathcal{M}}(x,x')$ is the Feynman Green function for the particular manifold, $\mathcal{M}$, and boundary conditions we are interested in. For flat space the geodesic parallel propagator, $g_{\mu\nu'}$, is just $\eta_{\mu\nu}\delta^{\nu}_{\nu'}$ and $R_{\mu\nu}$ is zero.

The parameter ξ in (1) is included for generality. If it is zero, then (1) gives the vacuum average of the canonical tensor, while if $\xi=\frac{1}{6}$ we get that of the improved tensor. For periodic boundary conditions these two cases give the same answer.

The improved tensor is automatically traceless in virtue of the equations of motion, $(\nabla_\mu\nabla^\mu+\xi R)\phi=0$ for the ϕ field and so we would use this as the best analogue of the electromagnetic case (DeWitt 1975).

For the general theory of the Casimir effect we are by no means restricted to the particular geometry of parallel plates. However it will be as well to consider this case in order to compare with known results.

The Green function for periodic conditions is

$$D_L(x,x')=\sum_{n=-\infty}^{\infty}D_F(\sigma_n(x,x')),\qquad x,x'\in L \tag{2}$$

where D_F is the usual massless Feynman Green function,

$$D_F(\sigma)=\frac{\mathrm{i}}{4\pi^2}\frac{1}{\sigma^2-\mathrm{i}\epsilon}$$

and σ_n is given by

$$\sigma_n^2(x,x')=(t-t')^2-(\boldsymbol{r}-\boldsymbol{r}_n')^2$$

with

$$x=(t,\boldsymbol{r}),\qquad x'=(t',\boldsymbol{r}')\qquad \boldsymbol{r}_n'=(x',y',z'+nL).$$

L is the distance between the plates and also stands for the manifold itself.

For (2) it is clear that $\nabla_{\mu'}D_L=-\nabla_\mu D_L\delta^\mu_{\mu'}$ and then from (1) we easily find for any ξ,

$$\langle T_{\mu\nu}(x)\rangle=\mathrm{i}\lim_{x'\to x}\nabla_\mu\nabla_\nu D_L,$$

using the equation of motion and the special conditions for flat space–time.

Then

$$\langle T_{\mu\nu}\rangle=\mathrm{i}\lim_{x'\to x}\sum_n\left(D_L''(\sigma_n)\nabla_\mu\sigma_n^2\nabla_\nu\sigma_n^2+D_L'(\sigma_n)\nabla_\mu\nabla_\nu\sigma_n^2\right) \tag{3}$$

with

$$D_L'\equiv \mathrm{d}D_L/\mathrm{d}\sigma^2.$$

We now try to effect the coincidence limit in (3). It is obvious that the $n=0$ term gives the infinite Minkowski result and hence the Casimir renormalization consists here simply of dropping this term. Use of the coincidence limits

$$\lim_{x'\to x}\begin{cases}\sigma_n^2(x,x')=-n^2L^2\\ \nabla_\mu\sigma_n^2(x,x')=2nL\delta_{\mu3}\\ \nabla_\mu\nabla_\nu\sigma_n^2(x,x')=2g_{\mu\nu}\end{cases}$$

gives for the remainder of the sum,

$$\langle T_{\mu\nu}\rangle_{\mathrm{C}}=-2^4\frac{\pi^2B_4}{2L^44!}\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&3\end{pmatrix} \tag{4}$$

Apart from a factor of 2^4, which can be attributed to the use of periodic boundary conditions, the value (4) is half Casimir's as expected (see DeWitt 1975).

This calculation is transparently invariant and is, of course, quite equivalent to the usual one. The eigenvalue sum has been replaced by an image one. There is also no need for any formal regularization since the infinite space contribution is readily picked out. However, we could have used, say, dimensional regularization methods to obtain the same result and this shows that covariant regularization methods are perfectly acceptable. This is not surprising.

The place where the usual covariance arguments go wrong is in the first step. Covariance does not, in general, require that $\langle T_{\mu\nu}\rangle$ be proportional to the Minkowski metric $\eta_{\mu\nu}$. This is true only if the symmetry group *in the large* of the manifold $\mathscr{M}$ is the Poincaré group. Any modification of the topological properties of infinite flat space–time will, in general, destroy global Poincaré invariance. Of course it will still exist as a local invariance (the metric is still $\eta_{\mu\nu}$) but this is not enough.

All that this is saying is the very obvious fact that the structure of $\langle T_{\mu\nu}\rangle$ will be determined by the global geometry of $\mathscr{M}$, and by the dynamics as well, naturally. A covariance argument along these lines is given by DeWitt (1975, p 306) for the present situation.

On a more technical level if we had divided space into rectangular solids, rather than slabs, the answer would have been

$$\langle T_{00}\rangle_{\mathrm{C}} = -(2\pi)^{-2} \sum_{\{n_1,\, n_2,\, n_3\}}{}' (L_1^2 n_1^2 + L_2^2 n_2^2 + L_3^2 n_3^2)^{-2}. \tag{5}$$

Not surprisingly we need Ewald lattice sums. A case which can be done 'exactly' is $L_1 = \infty, L_2 = L_3 = L$. Then

$$\langle T_{00}\rangle_{\mathrm{C}} = -(2\pi)^{-2} L^{-4} \sum_{\{n_2, n_3\}}{}' (n_2^2 + n_3^2)^{-2} = -\tfrac{1}{3} L^{-4} \beta(2),$$

where $\beta(2)$ is Catalan's constant.

These results are just special cases of the more general situation where $\mathscr{M}$ is a homogeneous space of infinite Minkowski space, $\tilde{\mathscr{M}}$. More precisely, if we consider periodicity in the time direction as unphysical we could take the spatial section, ${}^{(3)}\mathscr{M}$, of $\mathscr{M}$ to be one of the flat-space homogeneous forms obtained from the spatial sections, ${}^{(3)}\tilde{\mathscr{M}}$ of infinite Minkowski space by identifying points equivalent under a finite subgroup, Γ, of the Euclidean group E(3), i.e. ${}^{(3)}\mathscr{M} = {}^{(3)}\tilde{\mathscr{M}}/\Gamma$. These subgroups have been classified and pictured by Hantzsche and Wendt (1935). It is not expected that one would learn anything more physically about the Casimir effect by studying these different static geometries.

A different problem, but still only a technical one, is raised if one demands homogeneous boundary conditions instead of periodic ones. For the slab case, the Green function for ϕ (or $\partial\phi/\partial x$) zero on the boundaries is given by the method of images (e.g., Sommerfeld 1949) as

$$D_L^H(x, x') = D_{2L}(x, x') \mp D_{2L}(x, x'_R), \qquad \begin{cases} -\text{for } \phi = 0 \\ +\text{for } \partial\phi/\partial x = 0 \end{cases} \tag{6}$$

where $x_R = (t, x, y, -z)$ and D_{2L} is given by (2) with L replaced by $2L$.

The contribution to $\langle T^{\mu\nu}\rangle_{\mathrm{C}}$ from the first term of (6) is just the previous value, (4), divided by 2^4, while that from the second term is zero, in agreement with the results given by DeWitt (1975). If it did not give zero it would produce a z dependence in $\langle T^{\mu\nu}\rangle_C$.

The method of images can be extended to rectangular cavities. We would again expect that only the leading Green function would contribute to the improved $\langle T^{\mu\nu}\rangle_C$ to yield a value of 2^4 smaller than (5). This appears to disagree numerically with a result of Onley (1973).

Other cavity shapes can be treated by quite elegant methods but this will not be pursued here.

3. Casimir effect in curved space

It has been suggested (DeWitt 1975) that if the spatial section of space–time is closed (we have in mind $\epsilon = 1$ Robertson–Walker (RW) metrics) then there should be a Casimir term in the renormalized $\langle T_{\mu\nu}\rangle$.

This expectation is based on the result of the previous section. There, space was compact and had the topology $S^1 \times R^2$ for the periodic slab and T^3 for the periodic box.

Ford (1975) has performed the analogue of Casimir's calculation for the case of the Einstein universe and, thence, by conformal transformation, all spherical RW spaces. The calculation is a non-covariant one and it will be useful if we now give an invariant treatment along the lines of that in § 2, and see how far we can reproduce Ford's results.

The massless Green function in an Einstein universe can be written in various ways but the most appropriate one for us is an expression in terms of a sum over classical paths on the spatial section, S^3, of space–time (Dowker 1971). Thus we have

$$D_F(x, x') = \frac{\mathrm{i}}{4\pi^2 a} \sum_{n=-\infty}^{\infty} \frac{s + 2\pi n a}{\sin (s/a)} [(t-t')^2 - (s+2\pi n a)^2 - \mathrm{i}\epsilon]^{-1}. \tag{7}$$

The geodesic distance on S^3 is denoted by $s(q, q')$; $q, q' \in S^3$ and $x = (t, q)$.

Expression (7) is a sum of elementary solutions of the wave equation and an invariant regularization will consist of dropping the $n = 0$ term. This is the only term that is infinite at the coincidence limit, $t = t'$, $s = 0$, and is the only term to survive as the radius, a, tends to infinity to give the Minkowski expression.

Equation (7), less the $n = 0$ term, is substituted into (1) which is easily evaluated using the results

$$\lim_{x'\to x} \begin{cases} \nabla_0\nabla_0 D_{\text{ren}} = -\nabla_0\nabla_{0'} D_{\text{ren}} = -\dfrac{3\mathrm{i}}{16\pi^6 a^4} \displaystyle\sum_1^{\infty} \frac{1}{n^4} \\ \nabla_j\nabla_i D_{\text{ren}} = -\nabla_{j'}\nabla_i D_{\text{ren}} = -\dfrac{\mathrm{i}}{8\pi^4 a^4} g_{ij} \displaystyle\sum_1^{\infty} \left(\frac{1}{3n^2} + \frac{1}{2\pi^2 n^4}\right) \\ D_{\text{ren}} = -\dfrac{\mathrm{i}}{8\pi^4 a^2} \displaystyle\sum_1^{\infty} \frac{1}{n^2} \end{cases}$$

to give Ford's value,

$$\langle T_\mu^{\nu}\rangle_C = \frac{3}{16 a^4 \pi^6} \sum_1^{\infty} \frac{1}{n^4} \begin{pmatrix} 1 & & & \\ & -\frac{1}{3} & & \\ & & -\frac{1}{3} & \\ & & & -\frac{1}{3} \end{pmatrix} = \frac{1}{480\pi^2 a^4} (\delta_\mu^{\cdot\nu} - \tfrac{2}{3} a^2 R_\mu^{\nu}). \tag{8}$$

The neutrino result is similar.

Again, this is a completely invariant calculation and it is clear that any invariant method of regularization would yield the same answer. There would therefore be no need, other than that of curiosity, to develop the dimensional regularization expressions in this case.

The next step of Ford's argument involves a conformal transformation and it is here that possible problems arise. To see why, consider De Sitter space S_4^1 which can be reached from the Einstein universe by a conformal transformation. In this case it is just as easy to work in S_4^1 from the beginning. We can then compare answers. In fact there is no need to perform any calculations. S_4^1 possesses a ten parameter global group of motions, the De Sitter group, which contracts fairly nicely to the Poincaré group. Covariance arguments thus parallel those in infinite Minkowski space and we can write

$$\langle T_{\mu\nu}\rangle \propto g_{\mu\nu}, \qquad \langle T_{\mu\nu}\rangle_C \propto g_{\mu\nu},$$

which, together with the traceless condition of $\langle T_\mu^\nu\rangle$, seem to rule out any non-zero vacuum energy. Whence, then, comes the Casimir term calculated by Ford?

Firstly, it is obvious that Ford's calculation must destroy at least global De Sitter invariance at some point. (This may not be unwelcome).

As is well known, under the conformal rescaling, $g^{\mu\nu} \to \Omega g^{\mu\nu} = \tilde{g}^{\mu\nu}$, the Green function becomes,

$$D_F(x, x') \to \tilde{D}_F(x, x') = \Omega^{1/2}(x)\Omega^{1/2}(x')D_F(x, x'), \tag{9}$$

and the energy–momentum tensor changes as (Parker 1973)

$$T_\mu^\nu \to \tilde{T}_\mu^\nu = \Omega^2 T_\mu^\nu, \qquad \text{or} \qquad \tilde{T}_{\mu\nu} = \Omega T_{\mu\nu}. \tag{10}$$

It is equation (10) that Ford effectively uses to obtain $\langle \tilde{T}_{\mu\nu}\rangle$ in the conformally Einstein RW metric, $\tilde{g}_{\mu\nu}$, from that, $\langle T_{\mu\nu}\rangle$, in the Einstein one, $g_{\mu\nu}$. In this case the conformal factor $\Omega^{-1/2}$ is the radius function, R, and if we transform $\tilde{T}_\mu^\nu$ back to co-moving RW coordinates we find

$$\tilde{T}_{00} = \Omega^2 T_{00}, \qquad \tilde{T}_{ij} = \Omega T_{ij}, \tag{11}$$

where $T_{\mu\nu}$ is the energy–momentum tensor in an Einstein universe of unit spatial radius.

Ford's result follows if we take the vacuum average of (10), or (11), considered as operator equations, and use (8) for the average on the right-hand side. Thus, e.g.,

$$\langle \tilde{T}_{00}\rangle_C = \frac{3}{16\pi^6 R^4}\sum_1^\infty \frac{1}{n^4}.$$

This conclusion is not covariant. In order to see why we restate the calculation as follows.

Formally, if $\langle \tilde{T}_{\mu\nu}\rangle$ and $\langle T_{\mu\nu}\rangle$ are calculated, using (1), from the corresponding $\tilde{D}_F$ and D_F, then we will certainly find that they are related by (10),

$$\langle \tilde{T}_\mu^\nu\rangle = \Omega^2\langle T_\mu^\nu\rangle. \tag{12}$$

This equation holds before renormalization but *not* afterwards, in the sense that, if we use the Einstein renormalized $\langle T_{\mu\nu}\rangle_C$ (8), on the right, the resulting renormalized $\langle \tilde{T}_{\mu\nu}\rangle_C$ will not be the same as that which would have been produced by a regularization procedure covariant in the RW space.

Saying it another way, if we renormalize $\tilde{D}_F$ of (9) by dropping the $n = 0$ term in the expansion of D_F, (7), we will not obtain a covariantly renormalized object. In particular, it is easy to see that for De Sitter space the renormalized Green function, and hence the renormalized $\langle \tilde{T}_{\mu\nu} \rangle_C$ will not be invariant under the De Sitter group. This, we believe, is the reason for the difference between Ford's result and the invariant, zero one. We might say, in this case, that the operations of renormalization and conformal scaling do not commute.

In the general case therefore it will be necessary to use a generally covariant method of regularization. This agrees with the conclusions of Fulling and Davies (1976), in two dimensions who advocate the use of covariant point splitting. Also Streeruwitz (1975) has performed what looks like a respectably covariant calculation of the vacuum energy in RW spaces.

4. Discussion and conclusion

It is not clear to us yet where the possible non-invariance of the vacuum under conformal scalings enters the picture. According to Fulling and Davies (1975), apparently the conformal properties enjoyed by the classical theory are not carried through into the quantum version—this is the so-called 'conformal anomaly'. This has also been noticed by Fronsdal (1975) in anti-De Sitter space. However, against this must be set the theory of Chernikov and Tagirov (1968) who fix a unique vacuum in De Sitter space precisely by requiring it to be conformally invariant and it is easily checked that equation (9) gives, with (7), that Feynman Green function corresponding to just this vacuum. There seems to be a puzzle here. Another property that must be incorporated into the discussion is that in De Sitter space it does not seem possible to define a global quantum Hamiltonian (Nachtmann 1967). This is because complete De Sitter invariance implies the existence of a symmetry transformation in the component of the identity of the De Sitter group that anticommutes with the generator of local time translations.

For this reason we do not wish to imply that Ford's analysis and results are physically wrong.

Finally, we have discovered that the image method of deriving the Casimir energy given in § 2 has already been used by Brown and Maclay (1969).

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